MATH 102:107, CLASS 20 (MON OCT 23) - RELATED RATES

(1) A circular bacterial colony has radius $r(t)$ and area $A(t)$ (so that $A(t) = \pi(r(t))^2$).

(a) Suppose that $r(t)$ grows at a constant rate of 2. Calculate $\frac{dA}{dt}$ when $r = 5$.

Solution: The following equation relates the radius and the area.

$$A = \pi r^2$$

If we take the derivative of both sides with respect to t, we get $A'(t) = 2\pi r(t) r'(t)$, or written another way,

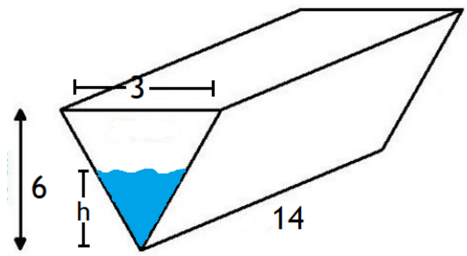
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We are told that $\frac{dr}{dt} = 2$ and $r = 5$. Then $\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$.

(b) Suppose instead that $A(t)$ grows at a constant rate of 20π. Calculate $\frac{dr}{dt}$ when $r = 5$.

Solution: We can use the same equation $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. We are given that $\frac{dA}{dt} = 20\pi$ and $r = 5$, and we want to calculate $\frac{dr}{dt}$. Therefore, $20\pi = 2\pi(5)\frac{dr}{dt}$ which means $\frac{dr}{dt} = 2$.

A water trough in the shape of a triangular prism has the dimensions shown (in dm). Let $h(t)$ denote the height of the water in the trough at time t, and let $V(t)$ denote the volume of the water.



(2) (a) Write an equation relating V and h.

Solution: The volume of water is

$$V = \frac{1}{2}(\text{height})(\text{width})(\text{length})$$

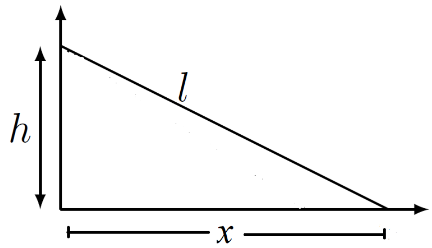
$$= \frac{1}{2}h \cdot (\text{width})(14)$$

The width of the water satisfies the equation $\frac{\text{width}}{h} = \frac{3}{6}$, by similar triangles - therefore the volume is $\boxed{V = \frac{7}{2}h^2}$.

(b) Water flows out at a constant rate of 2 dm^3/s. Find $\frac{dh}{dt}$, as a function of h.

Solution: Take the derivative of the boxed equation above to get $\frac{dV}{dt} = 7h\frac{dh}{dt}$. We are given that $\frac{dV}{dt} = -2$, so this means that $\boxed{\frac{dh}{dt} = -\frac{2}{7h}}$.

(3) A spider hops down from a point at height h on a wall, and starts walking along the ground away from the wall at a constant rate k. A thread of silk connects the point up on the wall to the spider's butt, and as the spider walks, the silk thread gets longer.



Let $l(t)$ denote the length of the thread of silk. Calculate $\frac{dl}{dt}$ when the spider is distance x from the wall.

Solution 1: The relationship between the length of the thread ($l(t)$) and the distance from the wall ($x(t)$) is summed up in the equation

$$l^2 = x^2 + h^2$$

Take the derivative of both sides to get

$$2l\frac{dl}{dt} = 2x\frac{dx}{dt}$$

(h is a constant that does not change with time.) Therefore, $\frac{dl}{dt} = \frac{x}{l}\frac{dx}{dt} = \frac{x}{\sqrt{x^2+h^2}}\frac{dx}{dt}$. If $\frac{dx}{dt} = k$, then this equals $\frac{kx}{\sqrt{x^2+h^2}}$.

Solution 2: Isolate l in the equation $l^2 = x^2 + h^2$ to get $l = \sqrt{x^2+h^2}$. Take the derivative of both sides with respect to time to get

$$\frac{dl}{dt} = \frac{1}{2\sqrt{x^2+h^2}}\left(2x\frac{dx}{dt}\right) = \frac{x}{\sqrt{x^2+h^2}}\frac{dx}{dt}$$

(remember that h is a constant which does not change with time.) Since $\frac{dx}{dt} = k$, we have $\boxed{\frac{dl}{dt} = \frac{kx}{\sqrt{x^2+h^2}}}$.

(4) A conical cup with height 10cm and radius 3cm is leaking water at a rate of 1ml/sec. When the water is at height h, calculate $\frac{dh}{dt}$. (The volume of a cone with radius R and height H is $\frac{\pi R^2 H}{3}$.)

Solution:: Let V be the volume of water in the cup. We are given that $\frac{dV}{dt} = -1$, and we want to calculate $\frac{dh}{dt}$ - therefore, we should find an equation relating V and h. Let r be the radius of the water in the cup. Then

$$V = \frac{\pi r^2 h}{3} \qquad \text{and} \qquad \frac{r}{h} = \frac{3}{10}$$

the second equation is by similar triangles. The second equation gives $r = \frac{3}{10}h$, plugging this back into the first equation gives

$$V = \frac{3\pi}{100}h^3$$

Taking the derivative gives

$$\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt}$$

We are given that $\frac{dV}{dt} = -1$, and therefore, $\boxed{\frac{dh}{dt} = -\frac{100}{9\pi h^2}}$.